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# Supersymmetry and the Yang-Mills effective action at finite N

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Abstract: We study the effective action of quantum mechanical SU($N$) Yang-Mills theories with sixteen supersymmetries and $N > 2$. We show that supersymmetry requires that the eight fermion terms in the supersymmetric completion of the $v^4$ terms be one-loop exact. We also show that the twelve fermion terms in the supersymmetric completion of the $v^6$ terms are two-loop exact for $N = 3$. For $N > 3$, this no longer seems to be true; we are able to find non-renormalization theorems for only certain twelve fermion structures. We call these structures 'generalized F-terms.' We argue that as the rank of the gauge group is increased, there can be more generalized F-terms at higher orders in the derivative expansion.

Keywords: Nonperturbative Effects, M(atrix) Theories, Extended Supersymmetry, Supersymmetric Effective Theories.

## Contents



# 1. Introduction

Symmetry principles give us rather remarkable control on the low-energy physics of supersymmetric gauge and gravity theories. For example, instanton corrections to the $F^4$ terms in three-dimensional N=8 SU(2) Yang-Mills can be determined using just supersymmetry [1]. Likewise, all D-instanton corrections to the $R^4$ terms in the type IIB string effective action are determined by supersymmetry [2, 3]. The aim of this work is to explore the extent to which supersymmetry determines the structure of the effective action of Yang-Mills theories with maximal supersymmetry and gauge group $\mathrm{SU}(N)$[1] with $N > 2$. In attempting to extend the technique developed in [4, 5] beyond rank one, we will find some interesting new physics and some subtleties.

We will study the effective action at a generic point on the Coulomb branch where the gauge group is broken to its maximal torus.[2] Largely for notational simplicity, we will study the quantum mechanical gauge theory that describes the low-energy dynamics of D0-branes [6, 7]. This theory first appeared in [8, 9]. A similar analysis can be performed for Yang-Mills theories in higher dimensions.

In section two, we consider terms of order $v^4$. More precisely, we study all possible eight fermion terms in the supersymmetric completion of $v^4$. We show

[1] It is worth pointing out that most of the arguments that we will use do not depend on the Weyl group of $\mathrm{SU}(N)$. The results should therefore extend to any group with rank $N-1$.

[2] It would be very interesting to perform a similar analysis for the effective action at a singularity where part of the non-abelian gauge group remains unbroken.

that all eight fermion terms must be generated at one-loop. Our results extend the analysis presented in [4],[10] for the quantum mechanical gauge theory. To show that the remaining terms at order $v^4$ are one-loop exact requires either constructing the full effective action using a Noether procedure, or using the arguments described in [11, 12]. We expect that all terms at order $v^4$ in the quantum mechanics are one-loop exact as a consequence of the non-renormalization of the eight fermion terms. This certainly is the case in four-dimensional Yang-Mills theory where all four derivative terms are generated at one-loop [13, 14, 15, 16]. Note that our results are in accord with expectations from Matrix theory [17].

In section three, we study the constraints imposed by supersymmetry on the twelve fermion terms in the supersymmetric completion of $v^6$. We show that these terms must be two-loop exact for SU(3). This agreement explains, in large part, why matrix theory correctly reproduces the three body interactions of supergravity to this order in the velocity expansion [18, 19, 20]. It is important to determine whether the twelve fermion terms completely determine the rest of the terms at order $v^6$ [21]. Hopefully, this can be determined using the kinds of arguments developed in [11, 12]. We suspect that this will be the case for $N = 2$ and $N = 3$, where the techniques described in section three might be sufficient to extend the arguments of [11, 12].

However, for higher rank gauge groups, we have not shown that the twelve fermion terms are two-loop exact. Rather, we are only able to show that special twelve fermion structures are protected by non-renormalization theorems. Supersymmetry does impose restrictions on the remaining possible twelve fermion structures. These restrictions should, for example, constrain the allowed tensor structures. However, it is not clear that the constraints are sufficient to completely determine the coupling constant dependence.

That certain twelve fermion terms might be renormalized beyond two-loops for $N > 3$ agrees with the conclusions reached in [22, 23] using perturbative arguments. Based on the results in [23], we would expect arbitrary renormalizations of certain $v^6$ structures for $N > 3$. In the final section, we describe the notion of generalized F-terms. We show that there are possible structures at order eight in the derivative expansion which, for $N > 3$, must be generalized F-terms. It seems likely that the terms found in [23] which agreed with supergravity are in the supersymmetric completion of generalized F-terms.

## 2. Constraining terms with four derivatives

### 2.1 Grading the eight fermion terms

The lagrangean for the supersymmetric quantum mechanics contains bosonic fields $x^i_A$ as well as fermions $\psi_{aA}$, where $i = 1, \ldots, 9$ and $a = 1, \ldots, 16$. The label $A = 1, \ldots, N-1$ denotes a particular element of the Cartan sub-algebra of the gauge group

$G$. The lagrangean describing the dynamics at a generic flat point has $\mathrm{Spin}(9) \times \mathcal{W}$ as a symmetry group, where $\mathcal{W}$ is the Weyl group of $G$. We will take $G$ to be $\mathrm{SU}(N)$.

The Spin(9) Clifford algebra can be represented by real symmetric matrices $\gamma^i_{ab}$, where $i = 1, \ldots, 9$ and $a = 1, \ldots, 16$. These matrices satisfy the relation

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}\,, \tag{2.1}$$

and a complete basis contains $\{I, \gamma^i, \gamma^{ij}, \gamma^{ijk}, \gamma^{ijkl}\}$, where we define:

$$\begin{aligned} \gamma^{ij} &= \frac{1}{2!}(\gamma^i\gamma^j - \gamma^j\gamma^i) \\ \gamma^{ijk} &= \frac{1}{3!}(\gamma^i\gamma^j\gamma^k - \gamma^j\gamma^i\gamma^k + \cdots) \\ \gamma^{ijkl} &= \frac{1}{4!}(\gamma^i\gamma^j\gamma^k\gamma^l - \gamma^j\gamma^i\gamma^k\gamma^l + \cdots)\,. \end{aligned} \tag{2.2}$$

The basis decomposes into symmetric $\{I, \gamma^i, \gamma^{ijkl}\}$, and antisymmetric matrices $\{\gamma^{ij}, \gamma^{ijk}\}$.

The lagrangean $L$ can be written as a sum of terms $L = \sum L_k$ where $L_k$ contains terms of order $2k$ in a derivative expansion. The order counts the number of derivatives plus twice the number of fermions. Supersymmetry requires the metric to be flat [4, 10]. The supersymmetry transformations then take the form

$$\begin{aligned} \delta x^i_A &= -i\epsilon\gamma^i\psi_A + \epsilon N^i_{AB}\psi_B \\ \delta\psi_{aA} &= (\gamma^i v^i_A \epsilon)_a + (M_A\epsilon)_a\,. \end{aligned} \tag{2.3}$$

The terms $N^i$ and $M$ encode all higher derivative corrections to the supersymmetry transformations and $\epsilon$ is a sixteen component Grassmann parameter. Note that once higher derivative terms appear in $L$, we must have $N^i$ and $M$ non-zero or the supersymmetry algebra no longer closes. Terms of order $v^4$ which appear in $L_2$ induce corrections to the lowest order supersymmetry transformations of order 2 in $N^i$ and order 3 in $M$. To determine the eight fermion terms, we will not need to know the detailed form of these corrections.

Following the argument for the SU(2) case [4], we can immediately conclude that the nine fermion terms which result by varying a boson in the eight fermion terms must vanish. We then obtain sixteen first order equations that must be satisfied by the eight fermion terms

$$\sum_{A,i,b} \gamma^i_{ab}\,\psi_{bA}\frac{\partial}{\partial x^i_A}\left(f^{(8)}(x)\right) = 0\,, \tag{2.4}$$

where we have schematically denoted the eight fermion terms by $f^{(8)}(x)$. Our task is then to unravel the extent to which (2.4) determines the eight fermion terms.

It is useful to grade the eight fermion terms in the following way: let us pick a preferred direction in the Cartan sub-algebra, say $A = 1$, with corresponding



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fermions $\psi_{a1}$. Any operator containing fermions can then be decomposed into pieces with fixed numbers of $\psi_{a1}$. We can then express the eight fermion terms in the form

$$f^{(8)}(x) = \sum_{i=0}^{8} f_i^{(8)}(x)\,, \tag{2.5}$$

where the eight fermion term $f_i^{(8)}(x)$ contains $i$ of our preferred fermions $\psi_{a1}$. Our constraint (2.4) implies

$$\left(\sum \gamma^i_{ab}\,\psi_{bA}\frac{\partial}{\partial x^i_A} f_8^{(8)}\right) + \left(\sum \gamma^i_{ab}\,\psi_{b1}\frac{\partial}{\partial x^i_1} f_7^{(8)}\right) = 0\,. \tag{2.6}$$

After multiplying (2.6) on the left by $\psi_{a1}$ and summing on $a$, we can conclude that

$$\left(\psi_1\gamma^i\psi_A\right)\frac{\partial}{\partial x^i_A} f_8^{(8)} = 0\,, \tag{2.7}$$

because the second term in (2.6) vanishes. Since $f_8^{(8)}$ only contains $\psi_1$ fermions, we obtain sixteen equations:

$$\sum_b \gamma^i_{ab}\psi_{b1}\frac{\partial}{\partial x^i_A} f_8^{(8)} = 0\,, \tag{2.8}$$

for every $A$. Note that the case $A = 1$ follows directly from (2.4).

## 2.2 The homogeneity of the eight fermion terms

We now want to show that $f_8^{(8)}$ is one-loop exact. The coupling constant, $g^2$, has mass dimension 3 in these quantum mechanical gauge theories. The fermions are dimension 3/2 while the scalars are dimension 1. If $f_8^{(8)}$ is one-loop exact, it must then be a homogeneous function of the scalars $x^i_A$ of degree $-11$. For example in the rank 1 case, some of the eight fermion terms had the form $g^2\psi^8/r^{11}$.

To show that this is again the case, let us apply $\gamma^j_{ca}x^j_A(\partial/\partial\psi_{c1})$ to (2.8) and sum on $a$:

$$\left(8\sum_i x^i_A\frac{\partial}{\partial x^i_A} - x^j_A\frac{\partial}{\partial x^i_A}\gamma^{ij}_{cb}\frac{\partial}{\partial\psi_{c1}}\psi_{b1}\right) f_8^{(8)} = 0\,. \tag{2.9}$$

Note that we have not yet summed on the $A$ index in (2.9). The second term in (2.9) contains operators that generate Spin(9) rotations on the bosons and fermions. Since $f^{(8)}$ is a term in the lagrangean, it is Spin(9) invariant. For $f_8^{(8)}$, this reduces to the assertion that

$$\sum_A\left(x^j_A\frac{\partial}{\partial x^i_A} - x^i_A\frac{\partial}{\partial x^j_A}\right) f_8^{(8)} = \frac{1}{2}\gamma^{ij}_{cb}\frac{\partial}{\partial\psi_{c1}}\psi_{b1} f_8^{(8)}. \tag{2.10}$$

We can use (2.10) to rewrite the second term in (2.9) after summing on $A$

$$\left(8r\frac{\partial}{\partial r} - \frac{1}{2}\sum_{i<j}\left(\gamma^{ij}_{cb}\frac{\partial}{\partial\psi_{c1}}\psi_{b1}\right)^2\right) f_8^{(8)} = 0\,, \tag{2.11}$$

where $r^2 = \sum_{i,A}(x^i_A)^2$.

The last term in (2.11) can be written as $2\rho_1(C)$ where $C$ denotes the Casimir operator of Spin(9) and $\rho_1$ denotes the representation of Spin(9) obtained from the product of eight $\psi_{c1}$ fermions. Since $f_8^{(8)}$ is invariant under (2.11), this representation must be a polynomial representation of Spin(9). Otherwise, we could not contract our eight fermions with scalars $x_A^i$ to get an invariant term. To determine the homogeneity of the $f_8^{(8)}$ term, we therefore need to evaluate the possible values of the Casimir appearing in (2.11).

### 2.3 Evaluating the Casimir

Let us introduce some notation for the weights of Spin(9). We will choose a Cartan sub-algebra, a Weyl chamber and an orthonormal basis of weight vectors $\langle w_1, w_2, w_3, w_4\rangle$ for the vector representation of Spin(9). The roots are constructed in terms of these weights and the positive roots are $w_i \pm w_j$ with $i < j$ and $w_i$. With this normalization, the sum of the positive roots, $2\delta$, is given by:

$$2\delta = 7w_1 + 5w_2 + 3w_3 + w_4\,.$$

The **16** spinor representation of Spin(9) then has highest weight

$$\frac{1}{2}(w_1 + w_2 + w_3 + w_4)\,,$$

and all the weights of this representation are of the form $\frac{1}{2}(\pm w_1 \pm w_2 \pm w_3 \pm w_4)$. The product of eight fermions is the reducible representation $\Lambda^8\,\mathbf{16}$. It is not hard to check that $\rho_1(C)$ takes its largest value on the irreducible sub-representation with highest weight $4w_1$.

In fact, we will see below that the constraint equations force $f_8^{(8)}$ to take values in this representation.[3] The value of the Casimir on an irreducible subspace of highest weight $\lambda$ is given by

$$\langle \lambda + 2\delta, \lambda\rangle\,,$$

and $2\rho_1(C)$ evaluated on $f_8^{(8)}$ then gives:

$$2\langle (7+4)w_1 + 5w_2 + 3w_3 + w_4, 4w_1\rangle = 88\,.$$

Equation (2.9) then becomes,

$$\left(r\frac{\partial}{\partial r} + 11\right) f_8^{(8)} = 0\,, \tag{2.12}$$

and the solution is homogeneous of degree $-11$ as claimed.

[3]Note that this representation contains the four scalar, two scalar and zero scalar terms that appeared in [4]. Together they form an irreducible representation of Spin(9).

We obtain weaker harmonicity constraints from (2.8) in the following way. Apply the operator

$$\gamma^q_{ac}\left(\frac{\partial}{\partial\psi_{c1}}\right)\left(\frac{\partial}{\partial x^q_A}\right),$$

to (2.8) and sum on $a$ to obtain,

$$\sum_i \frac{\partial^2}{(\partial x^i_A)^2} f^{(8)}_8 = 0\,, \tag{2.13}$$

for every $A$. Moreover this result is not dependent on the choice of coordinates used here. This means that $f^{(8)}_8$ is harmonic when restricted to any Spin(9) invariant 9-dimensional subspace of our $9r$-dimensional moduli space determined by a choice of element in the Cartan of SU($N$). Borrowing a term referring to a similar concept from the theory of several complex variables, we will call such functions pluri-harmonic.

We now show that $f^{(8)}_8$ must lie in the subspace with highest weight $4w_1$. This calculation will also be useful in our later analysis. We can choose coordinates for Spin(9) so that $\gamma^{12}$ is dual to the weight vector $w_1$. Since $\gamma^{12}$ squares to $-$I, we can decompose our fermions into eigenvectors of the $1-2$ generator of Spin(9) rotations on the fermions given in (2.10)

$$\psi_{aA} = \psi^+_{aA} + \psi^-_{aA}\,,$$

where $\psi^+_{aA}$ and $\psi^-_{aA}$ have eigenvalues $+i/2$ and $-i/2$, respectively. Note that $\psi^+_{aA}$ and $\psi^-_{aA}$ are complex conjugates.

Likewise, we can decompose the canonical momenta $p^i_A$ obeying the usual commutation relations

$$[x^i_A, p^j_B] = i\delta^{ij}\delta_{AB}\,,$$

into eigenvectors under the $1-2$ generator of Spin(9) rotations on the bosons also given in (2.10). In this case, $p^j_A$ for $j \neq 1,2$ is clearly annihilated by the rotation generator. The remaining two momenta are conveniently written as

$$\begin{aligned} p^1_A &= \tfrac{1}{2}\left(\partial_{z_A} + \partial_{\bar{z}_A}\right) \\ p^2_A &= -\tfrac{i}{2}\left(\partial_{z_A} - \partial_{\bar{z}_A}\right), \end{aligned} \tag{2.14}$$

where $\partial_{z_A}$ and $\partial_{\bar{z}_A}$ have eigenvalues $-i$ and $i$, respectively.

With these observations, we can decompose the free supercharge,[4]

$$Q_a = \gamma^i_{ab}\psi_{bA}\,p^i_A\,, \tag{2.15}$$

into a sum of two operators $Q^+_a + Q^-_a$, where $Q^+_a$ raises the $w_1$ component of the weight by $1/2$ and $Q^-_a$ lowers the $w_1$ component of the weight by $1/2$. Note that $Q^-_a$ is the

[4]Throughout this paper, when referring to the supercharge, we will mean the operator given in (2.15) that increases fermion number. We will never need the component that decreases fermion number.

complex conjugate of $Q_a^+$. We may further decompose $Q_a^-$ (and correspondingly $Q_a^+$) into a sum of two operators: one which raises the fermionic $w_1$ component of the weight by 1/2 and therefore lowers the bosonic component by 1, and one which leaves the bosonic component unchanged and lowers the fermionic component by 1/2. With a choice of complex coordinates, the first operator is simply $dz_{\alpha A}\partial_{z_A}$ where $\alpha$ runs from 1 to 8 and $dz_{\alpha A}$ is a linear combination of $\psi^+_{aA}$.

The highest weight component is automatically annihilated by $Q_a^+$. It must also be annihilated by the operator in $Q_a^-$,

$$dz_{\alpha 1}\partial_{z_1}\,,$$

for all $\alpha$. This implies that this highest weight component is either anti-holomorphic in the $z_1$ variable, or that it is annihilated by $dz_{\alpha 1}$ for all $\alpha$. The first condition is not possible.

To see this, note that the eight fermion term can only be singular at a point where non-abelian gauge symmetry is restored. These loci are codimension nine in the moduli space. So as we go off to infinity in almost all directions, the eight fermion term must vanish. However, any anti-holomorphic function that is bounded almost everywhere is constant. A constant eight fermion term is unphysical. We must therefore satisfy the second condition. This condition means that the highest weight component of $f_8^{(8)}$ is a multiple of $\prod_{\alpha=1}^{8} dz_{\alpha 1}$. This structure has weight $4w_1$ as we desired.

As a bonus, this argument shows that if $f_8^{(8)} = 0$ then $f^{(8)} = 0$, because the highest weight term with the greatest number of $\psi_{a1}$ factors must contain all 8 $dz_{\alpha 1}$ factors. Since $f_8^{(8)}$ determines all the remaining eight fermion terms, the remaining terms must all be one-loop exact. Therefore, the eight fermion terms are one-loop exact.

## 3. Constraining terms with six derivatives

### 3.1 The homogeneity of a special twelve fermion term

We can now consider the twelve fermion terms in the supersymmetric completion of $v^6$. As before, we can grade the twelve fermion terms according to the number $i$ of $\psi_{a1}$ factors:

$$f^{(12)} = \sum_{i=0}^{12} f_i^{(12)}\,. \tag{3.1}$$

Supersymmetry now requires that the thirteen fermion term obtained by varying $f^{(12)}$ satisfy the following equations for each $a$ [5],

$$\sum_{A,i,b} \gamma^i_{ab}\,\psi_{bA}\frac{\partial}{\partial x^i_A}\left(f^{(12)}(x)\right) = \delta_a L_2\,. \tag{3.2}$$

All that we need to know about the source terms $\delta_a L_2$ is that they are generated by varying terms of order $v^4$ contained in $L_2$ using corrections to the supersymmetry transformations, encoded in $N$ and $M$ of (2.3), generated by these terms of order $v^4$. The source terms in (3.2) are therefore two-loop exact, and the corresponding solution to (3.2) will be the sum of a two-loop exact term and a solution to the associated homogeneous equation

$$\sum_{A,i,b} \gamma^i_{ab}\, \psi_{bA} \frac{\partial}{\partial x^i_A} \left( f^{(12)}(x) \right) = 0\,. \tag{3.3}$$

We are then left to analyze the solutions of (3.3). In the rank one case, there was no solution to (3.3).[5] Is this again the case for a higher rank gauge group?

We begin as in the $f^{(8)}$ case by considering homogeneous $f^{(12)}_{12}$, which we henceforth assume is a solution to (3.3). The same argument as before gives

$$\sum_b \gamma^i_{ab} \psi_{b1} \frac{\partial}{\partial x^i_A} f^{(12)}_{12} = 0\,, \tag{3.4}$$

for every $a$ and $A$. Equation (3.4) again implies pluri-harmonicity, and retracing the argument for the eight fermion case, we again obtain a radial equation:

$$\left( 4r\frac{\partial}{\partial r} + 2\rho_1(C) \right) f^{(12)}_{12} = 0\,. \tag{3.5}$$

The only difference with the previous case is that the coefficient of $r\partial_r$ for an $f^{(k)}_k$ term is given by $16 - k$. We are left again with evaluating the Casimir term. The representation again must contain polynomial representations since we construct an invariant term by contracting our fermion structure with scalars $x^i_A$. The key question is determining an upper bound on the highest weight of the representations that can appear in the product of 12 fermions. It is easy to check that the highest weight that can appear in the exterior product of 12 fermions transforming in the **16** of Spin(9) is $2w_1$. So the largest value of the Casimir term is

$$2\rho_1(C) \leq 2(14+4) = 36\,.$$

This implies that $f^{(12)}_{12}$ has homogeneity $-9$ or larger. However, this scaling behavior corresponds to a negative power of the coupling constant and is therefore again ruled out. So at least the homogeneous solution for $f^{(12)}_{12}$ must vanish for any $N$.

### 3.2 Determining the remaining twelve fermion terms

Unlike the eight fermion case, it no longer follows readily that if $f^{(12)}_{12} = 0$, the rest of the $f^{(12)}_i$ terms must vanish. In fact, we have found that there are twelve fermion structures that can potentially be renormalized beyond two-loops for $N > 3$. For the

[5] There was a homogeneous solution to the weaker harmonicity equation. This solution required a negative power of the coupling constant and so was unphysical [5].

rest of this section, we will restrict to the $N = 3$ case where we will show that the twelve fermion terms cannot be renormalized beyond two-loops.

From the previous discussion, we know that the homogeneous solution for $f_{12}^{(12)}$ must vanish for any $N$. Let us take $f_{i_{max}}^{(12)}$ as the $f_i^{12}$ term with largest $i$ which is non-zero. What follows immediately from (3.2) with the source term zero is that $f_{i_{max}}^{(12)}$ is harmonic in the $x_1$ direction. From (3.2), we can still deduce that

$$\left(\psi_1 \gamma^i \psi_A\right) \frac{\partial}{\partial x_A^i} f_{i_{max}}^{(12)} = 0\,, \tag{3.6}$$

but this equation no longer implies a relation analogous to (3.4).

We will therefore have to use a different strategy to constrain the remaining possible twelve fermion terms. Since $f_{i_{max}}^{(12)}$ is killed by $\Delta_1$, we can reduce $i_{max}$ by applying $\Delta_1$ to $f^{(12)}$. Moreover, this 'new' twelve fermion term is still killed by each supercharge $Q_a$. Now applying a Laplacian to the twelve fermion term only makes it decay more quickly. So this modification decreases the homogeneity. This procedure should therefore give us a lower bound on the homogeneity of $f^{(12)}$. We can keep repeating this procedure until all terms in the resulting twelve fermion term are killed by $\Delta_1$.

Having applied this procedure in the $A = 1$ direction, we can repeat the process along all the other directions in the Cartan sub-algebra until the resulting twelve fermion term, let us call it $\widetilde{f}^{(12)}$, is pluri-harmonic. The second step involves operators $Q^*_{a1}$, where

$$Q^*_{aA} = \gamma^j_{as} \frac{\partial}{\partial \psi_{sA}} \frac{\partial}{\partial x_A^j}\,, \tag{3.7}$$

with no sum on $A$. These operators anti-commute with each other and reduce fermion number. On pluri-harmonic forms, these operators also anti-commute with $Q_b$.

Now we apply the operators $Q^*_{a1}$ and $Q^*_{a2}$ to $\widetilde{f}^{(12)}$ until we obtain a new $K$ fermion term $h^{(K)}$ which is killed by all of the operators $Q_a, Q^*_{aA}$, with respect to any choice of coordinates for the Cartan. This term is in general no longer invariant under Spin(9). By our earlier discussion, we see that $K \geq 8$. We expand $h^{(K)}$ as before:

$$h^{(K)} = \sum_i h_i^{(K)}\,.$$

We shall first show that $i \leq 8$. Note that $h_{i_{max}}^{(K)}$ is killed by

$$\gamma^j_{as} \psi_{s1} \frac{\partial}{\partial x_1^j}\,,$$

and by

$$\gamma^j_{as} \frac{\partial}{\partial \psi_{s1}} \frac{\partial}{\partial x_1^j}\,.$$

If we let $r_i = |x_i|$ then

$$\begin{aligned}
0 &= \left( x_1^k \gamma_{at}^k \frac{\partial}{\partial \psi_{t1}} \gamma_{as}^j \psi_{s1} \frac{\partial}{\partial x_1^j} - x_1^k \gamma_{at}^k \psi_{t1} \gamma_{as}^j \frac{\partial}{\partial \psi_{s1}} \frac{\partial}{\partial x_1^j} \right) h_{i_{max}}^{(K)} \\
&= \left[ \left( \frac{\partial}{\partial \psi_{s1}} \psi_{s1} - \psi_{s1} \frac{\partial}{\partial \psi_{s1}} \right) r_1 \frac{\partial}{\partial r_1} + \gamma_{ts}^{kj} \left( \frac{\partial}{\partial \psi_{t1}} \psi_{s1} - \frac{\partial}{\partial \psi_{t1}} \psi_{s1} \right) x_1^k \frac{\partial}{\partial x_1^j} \right] h_{i_{max}}^{(K)} \\
&= (16 - 2 i_{max}) r_1 \frac{\partial}{\partial r_1} h_{i_{max}}^{(K)} \,. \qquad (3.8)
\end{aligned}$$

So $h_{i_{max}}^{(K)}$ is constant in $r_1$ and therefore zero unless $i_{max} = 8$. We can apparently increase $i_{max}$ by acting on $h_8^{(K)}$ with

$$L = \psi_{s1} \frac{\partial}{\partial \psi_{s2}} \,.$$

This operation corresponds to an infinitesimal coordinate change in the Cartan. Since we have shown that $i_{max} = 8$ cannot in fact be increased, we can conclude that $L h_8^{(K)} = 0$. Anti-commuting $L$ and $Q_{a1}^*$ gives $\gamma_{as}^j \frac{\partial}{\partial \psi_{s2}} \frac{\partial}{\partial x_1^j}$, which must also kill $h_8^{(K)}$. It is easy to see that $h_8^{(K)}$ is then also killed by

$$x_1^k \gamma_{at}^k \psi_{t2} \gamma_{as}^j \frac{\partial}{\partial \psi_{s2}} \frac{\partial}{\partial x_1^j} = (K-8) r_1 \frac{\partial}{\partial r_1} + \gamma_{ts}^{kj} \psi_{t2} \frac{\partial}{\partial \psi_{s2}} x_1^k \frac{\partial}{\partial x_1^j} \,.$$

In a similar way, we can deduce from the relation $Q_{a2}^* h_{i_{max}}^{(K)} = 0$ that:

$$\left[ (K-8) r_2 \frac{\partial}{\partial r_2} + \gamma_{ts}^{kj} \psi_{t2} \frac{\partial}{\partial \psi_{s2}} x_2^k \frac{\partial}{\partial x_2^j} \right] h_8^{(K)} = 0 \,.$$

Combining these relations gives,

$$\left[ (K-8) r \frac{\partial}{\partial r} + \gamma_{ts}^{kj} \psi_{t2} \frac{\partial}{\partial \psi_{s2}} \left( x_1^k \frac{\partial}{\partial x_1^j} + x_2^k \frac{\partial}{\partial x_2^j} \right) \right] h_8^{(K)} = 0 \,. \qquad (3.9)$$

It is convenient to rewrite this relation in the following form

$$\left[ (K-8) r \frac{\partial}{\partial r} + (r_1 + r_2)(C) - r_1(C) + (s + r_2)(C) - s(C) \right] h_{8,s}^{(K)} = 0 \,. \qquad (3.10)$$

We define $s$ as follows: applying the $Q_{aE}^*$ operators sent the Spin(9) invariant $f^{(12)}$ into $h^{(K)}$ which is generally not Spin(9) invariant. We can decompose $h^{(K)}$ into components which satisfy an equation of the form

$$x_E^i \frac{\partial}{\partial x_E^j} - x_E^j \frac{\partial}{\partial x_E^i} + \gamma_{ts}^{ij} \psi_{tE} \frac{\partial}{\partial \psi_{sE}} + s(v^{ij}) h_{i,s}^{(K)} = 0 \,.$$

Here $s$ is a representation of Spin(9) and the $v^{ij}$ are generators of Spin(9). For example, if $K = 11$ then $s$ is the spinor representation. When $K = 12 - c$, $s$ is an irreducible representation appearing in the $c^{th}$ power of the spinor representation.

We are left again with a Casimir term, $(r_1+r_2)(C)-r_1(C)+(s+r_2)(C)-s(C)$. From representation theory, we learn that on an irreducible representation with highest $r_j$ weight of $l_j$ and highest $s$ weight of $l_3$, this term is bounded above by

$$\begin{aligned}(l_1+l_2+2\delta,l_1+l_2)-(l_1+2\delta,l_1)+(l_2+l_3+2\delta,l_2+l_3)-(l_3+2\delta,l_3)=\\ =2(l_1,l_2)+2(l_2+2\delta,l_2)+2(l_3,l_2)\,.\end{aligned}\tag{3.11}$$

Our prior computations imply that $l_1=4w_1$. Now we can work case by case: when $K=12$, we have $l_3=0$ and $l_2\leq 2w_1+2w_2$. This gives a Casimir term no bigger than 80 which implies homogeneity $\geq -20$. This corresponds to a two-loop correction.

When $K=11$, $2l_2\leq 3w_1+3w_2+w_3+w_4$ and $2l_3\leq w_1+w_2+w_3+w_4$. Therefore the Casimir term is bounded by,

$$\begin{aligned}(4w_1,3w_1+3w_2+w_3+w_4)+\frac{1}{2}(17w_1+13w_2+7w_3+3w_4,3w_1+3w_2+w_3+w_4)+\\ +\frac{1}{2}(w_1+w_2+w_3+w_4,3w_1+3w_2+w_3+w_4)=66\,.\end{aligned}$$

This implies homogeneity $\geq -22$ for $h^{(11)}$ and therefore homogeneity $\geq -21$ for the corresponding part of the twelve fermion term $f^{(12)}$. If we assume, as is physically reasonable, that the effective action is analytic in the coupling constant then this term is again generated at two-loops at worst.

When $K=10$, $l_2\leq w_1+w_2+w_3$ and $l_3\leq w_1+w_2+w_3+w_4$. The Casimir term is now bounded by

$$(4w_1,2w_1+2w_2+2w_3)+(8w_1+6w_2+4w_3+w_4,2w_1+2w_2+2w_3)+2|w_1+w_2+w_3|^2=50.$$

This gives homogeneity $\geq -25$ for $h^{(10)}$ and $\geq -23$ for the corresponding part of $f^{(12)}$. This is a potential three-loop term.

When $K=9$, the Casimir term is bounded by 25. This implies homogeneity $\geq -22$ for the corresponding term in $f^{(12)}$. Again, this is at worst two-loop assuming analyticity in the coupling constant.

When $K=8$, we have the additional equations,

$$\gamma^j_{as}\psi_{s1}\frac{\partial}{\partial x^j_2}h^{(K)}=0\,.$$

In a familiar way, we can extract the following relation:

$$(8r\frac{\partial}{\partial r}-r_1(v^{kj})(r_1+s)(v^{kj}))h^{(K)}_{8,s}=0\,.$$

In this case, the Casimir term is bounded by

$$\begin{aligned}(r_1+s)(C)+r_1(C)-s(C)&\leq 2r_1(C)+2(r_1,s)\\ &=104\,.\end{aligned}$$

This gives homogeneity $-9$ for the corresponding term in $f^{(12)}$, which must then come with an unphysical negative power of the coupling constant.

The one problematic case is $K = 10$ which could be generated at three loops. We will now show that this potential three-loop term cannot occur. In obtaining the three-loop bound, we assumed that the original twelve fermion term was pluri-harmonic. If it were not pluri-harmonic, we would have to apply at least one Laplacian to get a pluri-harmonic term. If this were the case, then our homogeneity bound should be raised by 2 which gives at most a two-loop correction, assuming analyticity in the coupling.

So let us consider a pluri-harmonic, Weyl invariant twelve fermion term $f_i^{(12)}$. Since we are restricting to the $N = 3$ case, we have two scalar fields in the Cartan which we will label $x^j$ and $y^j$. We can expand $f_i^{(12)}$ in spherical harmonics as a sum of terms of the form:

$$x^I y^J |x|^{-7-2|I|} |y|^{-7-2|J|} .$$

The multi-indices $I$ and $J$ are to be contracted with indices on an appropriate twelve fermion structure. The constraint equations then imply the existence of a term in $f_{12-i}^{(12)}$ of the form $x^L y^M |x|^{-7-2|L|} |y|^{-7-2|M|}$, where $|L| = |I| + 12 - 2i$ and $|M| = |J| - 12 + 2i$. This implies that $|I| \geq 2i - 12$.

A three-loop term has homogeneity $-23$ and so $14+|I|+|J| = 23$, or $|I|+|J| = 9$. This equality is incompatible with the previous bound on $|I|$ when $i > 10$. So, we need only consider $i_{max} = 10, 9$ and $8$.

We will eliminate these possible terms in a way which can also be used to get *lower bounds* on loop corrections. Recall that $f_{i_{max}}^{(12)}$ is killed by the operators $\gamma^j_{as}\psi_{s1}\frac{\partial}{\partial x_1^j}$. We then have the weaker equation

$$\left[(16 - i_{max}) r_1 \frac{\partial}{\partial r_1} + \gamma^{kj}_{ts} \frac{\partial}{\partial \psi_{t1}} \psi_{s1} x_1^k \frac{\partial}{\partial x_1^j}\right] f_{i_{max}}^{(12)} = 0 . \tag{3.12}$$

We have not used this equation for $N > 2$ because our invariance condition does not allow us to compute the Casimir term in general. However, we can estimate it. Using Spin(9) invariance, we rewrite this equation in the form:

$$\left[(16 - i_{max}) r_1 \frac{\partial}{\partial r_1} + 2\rho_1(C) - \gamma^{kj}_{ts} \frac{\partial}{\partial \psi_{t1}} \psi_{s1} x_2^k \frac{\partial}{\partial x_2^j}\right] f_{i_{max}}^{(12)} = 0 .$$

On terms in $f_i^{(12)}$ of the form $x^I y^J |x|^{-7-2|I|} |y|^{-7-2|J|}$, the operator $\gamma^{kj}_{ts}\frac{\partial}{\partial\psi_{t1}}\psi_{s1}x_2^k\frac{\partial}{\partial x_2^j}$, is bounded above by $|J|(16-i)$. The first Casimir term, $2\rho_1(C)$ is bounded above by 88 for $i_{max} = 8$, 84 for $i_{max} = 9$ and 60 for $i_{max} = 10$. We therefore see that the $x_1$ homogeneity given by $-7-|I|$ is bounded below by $-11-|J|$ for $i_{max} = 8$, $-12-|J|$ for $i_{max} = 9$ and $-10 - |J|$ for $i_{max} = 10$. Using $|I| + |J| = 9$, $|I|$ is bounded above by $13-|I|$ for $i_{max} = 8$, $14-|I|$ for $i_{max} = 9$ and $12-|I|$ for $i_{max} = 10$. This implies $|I| \leq 7$. From this bound, we can conclude that $i_{max} = 10$ leads at most to a two-loop correction. Moreover, our constraints imply that for $i_{max} = 9$, $(|I|, |J|) = (7, 2)$ or $(6, 3)$. Weyl invariance then implies that there must also be inadmissible solutions of the form $(|I|, |J|) = (8, 1)$ or $(9, 0)$. So $i_{max} = 9$ leads to at most a two-loop correction. A similar argument shows that $i_{max} = 8$ also leads to at most a two-loop correction. Therefore for $N = 3$, the twelve fermion terms are two-loop exact.

## 4. Generalized F-terms

Our technique for finding supersymmetry constraints can be summarized as follows: consider the terms at a given order in the velocity expansion with the largest number of fermions. Let us denote these 'top forms' collectively by $f^{(p)}$. In the simplest case, $f^{(p)}$ only contains fermions and no spacetime derivatives. Using the lowest order free-particle supersymmetry transformations, $f^{(p)}$ varies into a piece with one additional fermion and a piece with one fewer fermion. In searching for constraints, we generally want to restrict our attention to the piece with one additional fermion.

With this restriction, we can think of the supercharges $Q_a$ as differential operators acting as

$$Q_a = \gamma^i_{ab} \psi_{bA} p^i_A \, . \tag{4.1}$$

The restriction (4.1) neglects the supersymmetry variation of the fermions. The supersymmetry constraints then follow from the sixteen equations

$$Q_a f^{(p)} = S_a \, . \tag{4.2}$$

The $S_a$ are source terms obtained by varying lower order terms in the lagrangean using corrections to the supersymmetry transformations, encoded in $N, M$ of (2.3). Typically, only variations of lower order top forms appear in $S_a$.

Our theory always contains at least sixteen fermions so we always have top forms in the supersymmetric completion of the $v^4$ and $v^6$ terms. As we have seen, the equations (4.2) are strong enough to completely determine the coupling constant dependence of the eight fermion terms for any $N$. For $N = 3$, the same is true for the twelve fermion terms but this is no longer clear for higher $N$. For higher $N$, we should expect that only certain terms at a given order in the velocity expansion will be constrained by (4.2). We gave an example of such a term in section 3.1.

Let us return momentarily to the SU(2) case as an example. The top form in the supersymmetric completion of $v^8$ is a sixteen fermion term. There is a unique structure that takes the form

$$f^{(16)} = f(r)\, \psi_{a_1} \cdots \psi_{a_{16}} \epsilon^{a_1 \cdots a_{16}} \, , \tag{4.3}$$

for some radial function $f$. It is easy to check that $f^{(16)}$ can be written in the form

$$f^{(16)} = \{Q_1, \cdots \{Q_{16}, g(r)\}\} \, , \tag{4.4}$$

for some $g(r)$, obtained by appropriately integrating $f(r)$. The brackets appearing in (4.4) should be viewed as graded commutators i.e. anti-commutators for two fermionic operators and commutators for everything else. Now the variation of (4.3) into a term with seventeen fermions automatically vanishes since we only have sixteen fermions. From (4.2), we find no constraint on the choice of $g(r)$.

Top forms that can be written in the form (4.4) are natural generalizations of the superspace notion of a D-term. While there is no useful notion of superspace for theories with sixteen supercharges, it is still meaningful to ask whether a term in the lagrangean — not necessarily a top form — can be written in the form (4.4). Top forms that are generalized D-terms in this sense are automatically killed by each $Q_a$. In searching for supersymmetry constraints, we want to quotient out by these trivial solutions in the usual cohomological sense. It is then natural to define the set of generalized F-terms as all terms that cannot be written in the form (4.4).[6] Since for SU(2) all top forms at order eight in the derivative expansion are clearly D-terms, we do not expect supersymmetry to impose any simple restrictions on terms of this order.

However for higher rank, we have more fermions so there can be generalized F-terms at higher orders in the derivative expansion of the most general effective action compatible with sixteen supersymmetries and the global Spin(9) symmetry. A strong indication of the existence of such terms is the agreement of certain interactions in Matrix theory with supergravity found in [23]. We would hope that there is a simple argument showing that the agreement is because of non-renormalization theorems. This is an important open question.

We will conclude our discussion by showing that there are possible generalized F-terms in the supersymmetric completion of $v^8$ terms for $N > 3$. The argument goes as follows: we can choose a basis for the gamma matrices $\gamma^i$ so that $\gamma^9$ is diagonal while the rest are of the form

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix},$$

where $A$ is some $8 \times 8$ matrix. So $p^i \gamma^i_{ab} \psi_b$ can contain the fermions

$$\psi_1, \ldots, \psi_8, \psi_a ,$$

for $a \geq 9$. We are suppressing the Cartan labels for the moment and just focusing on the Spin(9) indices. For $a < 9$, we see that $p^i \gamma^i_{ab} \psi_b$ can contain the fermions

$$\psi_a, \psi_9, \ldots, \psi_{16} .$$

So, for example, $\psi_1$ can only occur for nine choices of $a$. We could then consider a top form

$$\psi_{1A_1} \cdots \psi_{1A_{16}} ,$$

[6] We wish to thank E. Witten for suggesting this definition.

which is possible if the rank of the group is sixteen or larger. This term, which can appear as part of some Spin(9) invariant top form, clearly cannot be written as a generalized D-term of the form (4.4). We can find analogous examples in theories with lower rank. Let us take rank three: a sixteen fermion term with the following fermion content

$$(\psi_{1A_1}\psi_{1A_2}\psi_{1A_3}\psi_{2A_1}\psi_{2A_2}\psi_{2A_3}\cdots\psi_{5A_1}\psi_{5A_2}\psi_{5A_3})\times\psi_{6A_3},$$

cannot be part of a generalized D-term. Only fourteen $Q_a$ could possibly contribute $\psi_1,\ldots,\psi_6$ fermions but we need sixteen such fermions. It is not hard to construct more examples of this kind.


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